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Concerning Klein's Group of $(n+1)!$ n-ary Collineations.[*]

By Eliakim Hastings Moore.

1. To the statement that a set of $n+1$ real numbers

$$y_0, y_1, \cdots, y_n$$

determines a point, one may give six geometric interpretations (which are really of the nature of definitions), in that the numbers y_i are

(I) Cartesian point coördinates in real flat space R_{n+1} of $n+1$ dimensions;

(II_u, II_b) Homogeneous point coördinates in R_n;

(III) Supernumerary Cartesian point coördinates in R_n;

(IV_u, IV_b) Supernumerary homogeneous point coördinates in R_{n-1};

with the restriction (in II and IV) that not all the homogeneous coördinates y of a point are 0, and (in III and IV) that the sum of the supernumerary coördinates y of a point is 0, and with the understanding that the points

$$(y_0', \cdots, y_n'), \quad (y_0'', \cdots, y_n'')$$

are the same point if and only if

$$y_i'' = \rho y_i', \qquad (i = 0, 1, \cdots, n),$$

where the proportionality factor ρ is a real number:

(I, III): ρ is 1;

(II_u, IV_u): ρ is any positive number;

(II_b, IV_b): ρ is any non-zero number.

One speaks of the geometries of metric space, of unilateral and of bilateral projective space. The bilateral projective space is the ordinary projective space, while the unilateral projective space R_n is in effect the space of rays diverging from a point in metric space R_{n+1}.

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[*] Presented to the American Mathematical Society, Chicago section, December 30, 1898, and conveniently modified in the present writing.

The supernumerary coördinates $y_0, \ldots, y_n$ (III, IV) are most conveniently introduced in terms of $x_1, \ldots, x_n$ (I, II) by the formulas:

$$(n+1)\,y_i \equiv (n+1)\,x_i - \sum_{i=0}^{i=n} x_i, \qquad (i=0,\,1,\,\ldots,\,n),$$

where x_0 is permanently 0, so that in the x's one has the identities:

$$\sum_{i=0}^{i=n} y_i \equiv 0; \quad y_i - y_j \equiv x_i - x_j, \qquad (i,\,j=0,\,1,\,\ldots,\,n).$$

2. Denoting by $a_0, a_1, \ldots, a_n$ any permutation a of the $n+1$ indices $0, 1, \ldots, n$, we consider the collineation S_a which transforms the point $(y_0, \ldots, y_n)$ to the point $(y'_0, \ldots, y'_n)$, where

$$y_i = y'_{a_i}, \qquad (i=0,\,1,\,\ldots,\,n).$$

The $(n+1)!$ collineations S_a corresponding to the $(n+1)!$ permutations are distinct, and form a group $G_{(n+1)!}$ of collineations simply isomorphic to the symmetric group on $n+1$ letters. For the interpretation IV_b this is Klein's group* of collineations permuting in all ways $n+1$ independent points of R_{n-1}.

3. In my paper† on the cross-ratio group of $n!$ Cremona transformations of order $n-3$ in flat space of $n-3$ dimensions, there are contained general theorems† of considerable interest concerning the fixed points (real and complex) of the collineations of Klein's group.

4. It seems desirable to determine, for the six well-known groups $G_{(n+1)!}$ of §2, the corresponding partitions of the respective spaces $(R_{n+1}, R_n, R_n, R_{n-1})$ into $(n+1)!$ regions. The regions are to be simply connected and conjugate or equivalent under the group. And a point P, which is invariant under exactly d collineations, is to belong to exactly d regions which permute transitively under those collineations, and two distinct points of a region are never to be con-

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* Klein, "Ueber eine geometrische Repräsentation der Resolventen algebraischer Gleichungen" (Mathematische Annalen, vol. IV, pp. 346-358, 1871).

† American Journal of Mathematics, vol. XXII, p. 279.—Cf. §§2 and 5, and the closing paragraphs of the paper.

jugate under the group. Thus every region contains exactly one of every set of conjugate points.

5. As is usual in the geometric group theory (after Klein), we take one of the regions as the fundamental region and denote it by I. Then the collineation S_a throws the region I to a region S_a. If the region I is defined by a system $\sigma(y_0, y_1, \ldots, y_n)$ of conditions (equalities and inequalities) on the coördinates $y_0, y_1, \ldots y_n$ of its general point, it is obvious that the region S_a is defined by the system $\sigma(y_{a_0}, y_{a_1}, \ldots, y_{a_n})$ of conditions on the coördinates $y_0, \ldots, y_n$ of its general point.

6. For the cases I, III, II_u, IV_u one has obviously the following very simple partition:

The region S_a *contains all points* $(y_0, \ldots, y_n)$ *for which*

$$y_{a_0} \leqq y_{a_1} \leqq \cdots \leqq y_{a_n}.$$

7. For the cases II_b, IV_b the partition of §6 would assign a point in general to two regions, since the point is unchanged if its coördinates suffer a simultaneous change of sign. By a proper determination of the coördinates as to sign, we may, however, arrange to hold to the partition of §6 also for these cases.

Let a point be given by the various sets $(\rho\eta_0, \ldots, \rho\eta_n)$ of coördinates $(y_0, \ldots, y_n)$, where ρ is any real non-zero number. For the purpose of the partition of §6, we specify that ρ shall have such a sign that when the numbers $\eta_0, \ldots, \eta_n$ are arranged in order* of increasing algebraic magnitude,

$$\zeta_0, \ldots, \zeta_n, \qquad (\zeta_0 \leqq \zeta_1 \leqq \cdots \leqq \zeta_n),$$

the first of the expressions:

$$\zeta_0 + \zeta_n,\ \zeta_1 + \zeta_{n-1}, \ldots, \zeta_m + \zeta_{n-m}, \cdots \qquad (m = 0, 1, \ldots, [n/2]),$$

not zero, shall be negative.† This stipulation, it is important to notice, is inva-

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* In case of any equalities amongst the η's, this order is not quite definite; this fact does not, however, interfere with the uses made of the order.

† Thus the point

$$(y_0, y_1, \ldots, y_k, \ldots, y_n) = (-n, 1, \ldots, 1, \ldots, 1)$$

is in the fundamental region I.

riant under the collineations of the group. The stipulation fails only if

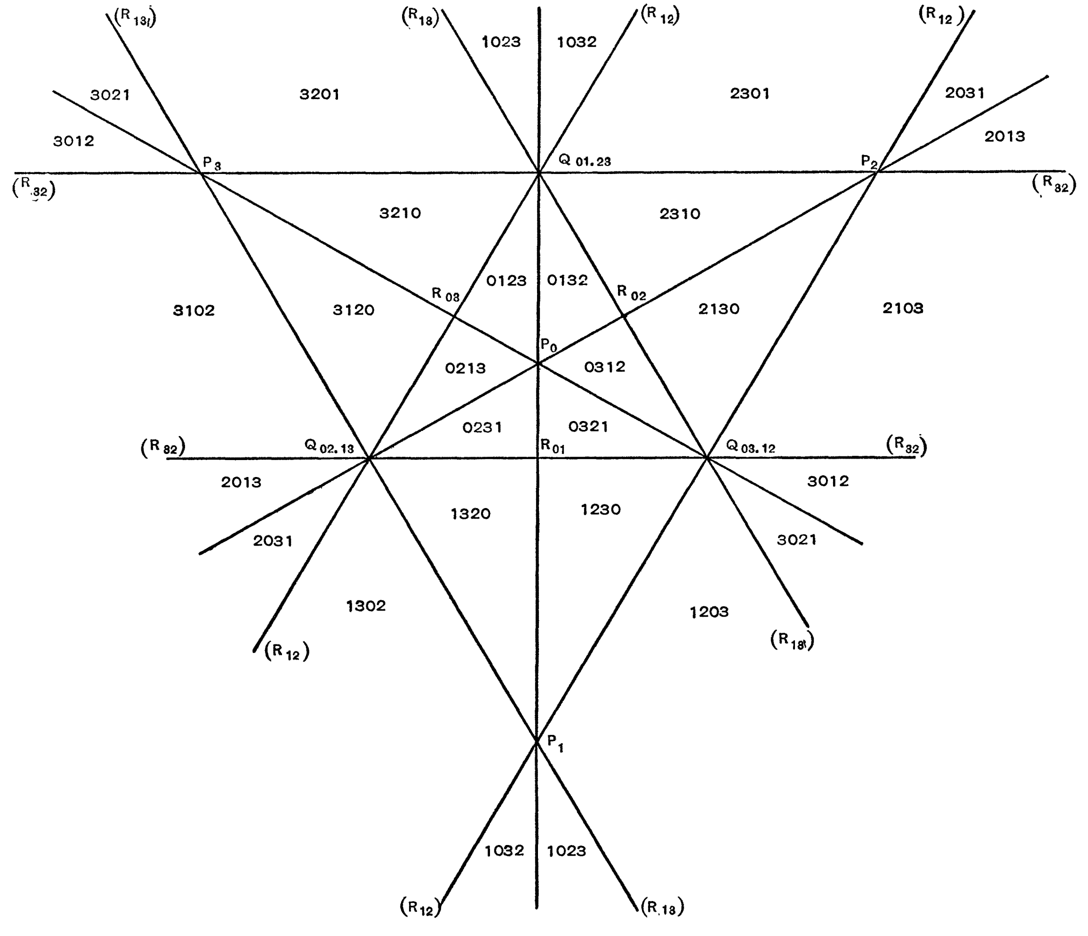
$$0=\zeta_0+\zeta_n=\zeta_1+\zeta_{n-1}=\,..\,=\zeta_m+\zeta_{n-m}=\,..\,,$$
$$(m=0,1,\,..\,,[n/2]).$$

In such a case, letting ρ have each sign, we assign by §6 the point $(\eta_0,\,..\,,\eta_n)$ to two regions; it is, indeed, invariant under the transformation interchanging the two regions, viz., the transformation interchanging its coördinates

$$\zeta_0,\zeta_n;\ \zeta_1,\zeta_{n-1};\,..\,;\ \zeta_m,\zeta_{n-m};\,..\,,$$

and accordingly it belongs in the two regions.

8. For all cases the partitions which were in view have been obtained. In illustration, I give the figure for the case IV_b: $n=3$. The 4 points $P_0\,P_1\,P_2\,P_3$



are permuted in 4! ways, and the 4! parts of the projective plane have as boundaries the 6 sides of the 4-gon $P_0 P_1 P_2 P_3$, and the 3 sides of its diagonal 3-gon $Q_{01.23}$, $Q_{02.31}$, $Q_{03.12}$. The x-coördinate triangle being $P_1 P_2 P_3$ and the x-unit point being P_0, the y-coördinate 4-side is the 4-side whose 6 vertices are the 6 simple intersections $R_{01}, \ldots, R_{34}$ of the 9 lines of the partition: R_{ij} lies on the lines $P_i P_j$, $Q_{ik.jl}\, Q_{il,jk}$, $y_k = 0$, $y_l = 0$, where $ijkl$ are the 4 indices 0123 in some order. The whole figure is a well-known fundamental figure of projective geometry. A convenient metrical specialization is given here.

The 4! regions $S_a = S_{a_0 a_1 a_2 a_3}$ are marked with the 4! permutations a. The fundamental region I is 0123 with the vertices $P_0\, Q_{01.23}\, R_{03}$. The region $S_{a_0 a_1 a_2 a_3}$ is the triangle $P_{a_0}\, Q_{a_0 a_1 . a_2 a_3}\, R_{a_0 a_3}$. The vertices P belong each to 6 regions, the vertices Q belong each to 8 regions, and the vertices R belong each to 4 regions.

The collineations

$$A_2 = S_{0213} = S_{(12)}, \quad A_3 = S_{0132} = S_{(23)}, \quad A = S_{3210} = S_{(03)(12)},$$

constitute a system of generators of the group; they are the projective reflections which throw the region I into the adjacent regions. From the figure in the vicinity of the region I one has the generational relations:

$$(\alpha) \qquad I = A_2^2 = A_3^2 = A^2 = (AA_2)^2 = (AA_3)^4 = (A_2 A_3)^3.$$

These relations, easily verified analytically, are not sufficient fully to characterize the group, for the relation:

$$(\beta) \qquad S_{3102}^3 = S_{(032)}^3 = (A_3 A_2 A)^3 = I,$$

is not derivable from them, since they are all even while it is odd in the elements $A_2 A_3 A$. We come back to this matter* in §10.

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* The following remarks are of interest. If we introduce

$$A_1 = S_{1023} = S_{(01)} = S_{(03)(12)}\, S_{(23)}\, S_{(03)(12)} = AA_3A,$$

we find that in the abstract group $G(A_2, A_3, A)(\alpha) = G(\alpha)$ generated by A_2, A_3, A, subject to the relations (α), the three elements $A_1 = AA_3A$, A_2, A_3 satisfy the standard relations:

$$(\gamma) \qquad I = A_1^2 = A_2^2 = A_3^2 = (A_1 A_2)^3 = (A_1 A_3)^2 = (A_2 A_3)^3$$

for the symmetric group $G_{4!}^4$ on 4 letters. (This fact was noticed by Mr. J. C. Hammond.) They generate a subgroup $G(A_1, A_2, A_3) = G_{4!}$ of the group G. That the group generated is not a subgroup of the $G_{4!}$ appears from consideration of the collineation group $G_{4!}$, which is a quotient group of G, and

9. In general the flat spaces

$$y_i - y_j = 0 \qquad (i \neq j;\ i, j = 0, 1, \ldots, n)$$

and, in cases II_b and IV_b,

$$y_i + y_j = 0 \qquad (i \neq j;\ i, j = 0, 1, \ldots, n)$$

effect the partition into the $(n+1)!$ regions. The space $y_i - y_j = 0$ is invariant point by point under the involutoric collineation interchanging the coördinates y_i, y_j—a transformation which reflects the partition into itself on the space, $y_i - y_j = 0$ interchanging the regions of the various pairs of regions lying on opposite sides of that space. The points of $y_i + y_j = 0$, however, apart from those lying in the boundaries just discussed, belong each to one region, with the exception of those satisfying relations of the type

$$0 = y_i + y_j = y_k + y_l = \ldots,$$

as was explained in §7. (Only for $n = 3$ of IV_b is $y_i + y_j = 0$ invariant point by point.) The space $y_i + y_j = 0$ is thus itself divided into regions, each of which belongs to one of the two fundamental regions of which it is a boundary.

10. One might study more closely the topological interrelations of the regions and their boundaries of various dimensions. In particular, by considering the fundamental region I and the regions S_g adjacent to it along boundaries of the highest dimension, one might (as usual*) determine a system of generators S_g of the group, and by means of the regions adjacent to it along boundaries of the highest two dimensions, obtain a system of grouptheoretic relations amongst these generators.

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which is, moreover, generated by A_1, A_2, A_3. On solving in the collineation group $G_{4!}$ for A in terms of A_1, A_2, A_3 one finds

$$A = A_1 A_2 A_3 A_2 A_1 A_2 .$$

In this on replacing A_1 by AA_3A one has a relation

$$A = AA_3AA_2A_3A_2AA_3AA_2,$$

which is equivalent under (α) to the relation (β). Thus the group $G(A_2, A_3, A)(\alpha, \beta)$ is its own group $G(A_1, A_2, A_3)$. Hence the relations (α, β) fully characterize the collineation group $G_{4!}$ as an abstract group; and, further, the group $G(\alpha)$ is the direct product of its subgroup $G(A_1, A_2, A_3)$ and the invariant subgroup H whose elements reduce to the identity by the adjunction of the relation (β).

*Cf. for example, Klein, "Modulfunctionen," I, p. 452, p. 455. Fricke u. Klein, "Automorphe Functionen," I, p. 168 fg.

With respect to any geometric group with fundamental region, it seems to be clear that the system of relations so obtained is a complete system of generational relations of the group in the sense† of Cayley and Dyck in case the fundamental region is simply connected and the set of all the equivalent regions forms a simply connected portion of all space.

These conditions are fulfilled here in cases I, III, and it was indeed in this way, from the geometric symmetric group of case I and its alternating subgroup that I obtained the abstract generational determination of the symmetric and alternating groups on n letters, which, in pure grouptheoretic form, I exhibited in the Proceedings of the London Mathematical Society, Dec. 10, 1896, vol. XXVIII, p. 357. One has the feeling that a generational determination of a group effected by geometric process is apt to be, as in the modular group and in the case just cited, of the utmost simplicity.

The conditions are, however, not fulfilled in cases II, IV, since projective spaces, whether bilateral or unilateral, are not simply connected; a closed straight line which passes through the infinite region of the plane cannot, by continuous deformation, shrink up to a point.—Thus, for example, in the case IV_b: $n=3$, considered in §8, the broken line corresponding to the relation (β) is closed through infinity.

THE UNIVERSITY OF CHICAGO, *September*, 1900.

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† Cayley, "The Theory of Groups," American Journal of Mathematics, vol. I, p. 50; Dyck, "Gruppentheoretische Studien," Mathematische Annalen, vol. XX, p. 1. Cf. also my paper: "On the Generational Determination of Abstract Groups," soon to be published in the Transactions of the American Mathematical Society.